

Zero range and finite range processes with asymmetric rate functions

Amit Kumar Chatterjee¹ and P. K. Mohanty^{1,2}

¹CMP Division, Saha Institute of Nuclear Physics, HBNI, 1/AF Bidhan Nagar, Kolkata, 700064 India.

²Max Planck Institute for the Physics of Complex Systems, N öthnitzer Straße 38, 01187 Dresden, Germany.

E-mail: amit.chatterjee@saha.ac.in; pk.mohanty@saha.ac.in

Abstract. We introduce and solve exactly a class of interacting particle systems in one dimension where particles hop asymmetrically. In its simplest form, namely asymmetric zero range process (AZRP), particles hop on a one dimensional periodic lattice with asymmetric hop rates; the rates for both right and left moves depend only on the occupation at the departure site but their *functional forms are different*. We show that AZRP leads to a factorized steady state (FSS) when its rate-functions satisfy certain constraints. We demonstrate with explicit examples that AZRP exhibits certain interesting features which were not possible in usual zero range process. Firstly, it can undergo a condensation transition depending on how often a particle makes a right move compared to a left one and secondly, the particle current in AZRP can reverse its direction as density is changed. We show that these features are common in other asymmetric models which have FSS, like the asymmetric misanthrope process where rate functions for right and left hops are different, and depend on occupation of both the departure and the arrival site. We also derive sufficient conditions for having cluster-factorized steady states for finite range process with such asymmetric rate functions and discuss possibility of condensation there.

Keywords: Zero-range processes, Non-equilibrium processes, Exact results

1. Introduction

Driven diffusive systems with stochastic dynamics have been studied extensively in recent years to understand macroscopic properties of non-equilibrium steady states [1]. Unlike stationary equilibrium systems which follow Gibbs measure, the non-equilibrium systems lead to unusual steady state measures with interesting nontrivial correlations, thermodynamic phases and phase transitions even in one dimension [2]. In absence of any generic method for obtaining exact steady state distributions in non-equilibrium systems, the analytical studies are limited to model systems where certain specific techniques like, pairwise balance [3], Bethe ansatz [4], matrix product ansatz [5] etc. can be applied. Zero range process (ZRP), introduced by Spitzer in [6] in context of invariant measures for interacting Markov processes, is one of the simplest model for which steady state is known exactly. In ZRP, particles hop - one at a time- to one of the nearest neighbors on a d -dimensional lattice with a specific rate function that depends *only* on the occupation of the departure site, justifying the name *zero range*. For any arbitrary form of the rate function, this model has a factorized steady state (FSS) [6]. In spite of its simplicity, ZRP exhibits a condensation transition for certain rate functions, even in one dimension ($1d$), where a macroscopic fraction of particles occupy a single site [7]. In $1d$, such a condensation transition can be mapped to a phase separation in one dimensional exclusion model with suitable diffusion dynamics. This mapping helps in identifying a generic criterion for having phase separation in one dimensional exclusion models [8]. The ZRP correspondence of exclusion models has been exploited to obtain spatial correlation functions in several systems [9].

A natural extension of ZRP is the finite range process (FRP), where the hop rate of the particles depends on the occupation number of not only the departure site but also that of all other sites within a specified distance [10]; clearly in FRP, particle-particle interaction extends to a finite number of neighboring lattice sites. A specific example, where the hop rate depends on occupation numbers of both the departure and the arrival sites, is commonly known as misanthrope process (MAP) [11]. Like ZRP, misanthrope process can also have factorized steady state [11], but only for a certain class of hop rates. However, factorized steady state is not possible for FRP [10] where three or more sites are involved in the hop rates. For these systems, in fact, one can obtain a cluster factorized steady state (CFSS) when the rates satisfy certain specific conditions [10]. The simplest example of a cluster factorized steady state is a pair factorized state, introduced by Evans *et. el.* [12], exhibiting condensation transition when particle interaction is tuned. Interestingly, unlike systems with factorized steady states (leading to a single site condensate), in FRP condensate can form over a extended region in the space [13] due to spatial correlations. Another interesting variation is ZRP with open boundaries where, in addition to the ZRP dynamics in the bulk, particles are allowed to enter or exit the system at the boundaries [14]. These open systems may not have well-defined stationary states for any arbitrary boundary dynamics, but condensation can occur for certain dynamics which lead to unique stationary measures

[14]. Recently, a non-markovian zero range process [15] is introduced to investigate the impact of temporal correlations on the dynamics of condensation.

Over years, zero range processes have found vast applications in different areas of science. It is being considered as a reasonably good model for mass transport processes [16] and sandpile dynamics [17, 18], reconstituting polymers [19] etc. Being an analytically tractable driven diffusive system, ZRP and related models have become a test ground for development of non-equilibrium thermodynamics [20]. These models also help in understanding experiments on shaken granular gases [21], dynamics of growing networks [22], aggregation of active filament bundles [23], wealth condensation [24], jamming in traffic flow [25], quantum gravity [26] etc. Due to their far reaching importance, ZRP and related models have found a significant place in the research activity in statistical mechanics (see Refs. [7, 11] for reviews).

In usual ZRP and related models, the hop rates do not depend on the direction along which the particles move. Although, recently some simple examples [27] have been studied in two dimension ($2d$), where the rate functions are different in x - and y - directions, but it was observed that the two point correlations are finite indicating that the steady state is not factorized. Later, a generalized zero range processes was introduced [28] where more than one particle can hop from a site and the hop rates may depend on direction of hopping. A sufficient condition for having FSS in these models, which is also conjectured as the necessary condition, showed explicitly that indeed models described in [27] cannot have factorized steady states. Moreover, these models in $1d$ (with one hop at a time) reduce to an asymmetric ZRP where particles hop to right or left neighbour with rates $u_R(n) = pu(n)$, $u_L(n) = qu(n)$ respectively; notably, the steady state weights of these models do not depend on p, q and the asymmetry parameter $\frac{p}{q}$ only redefines the fugacity of the system in grand canonical ensemble.

In this article we introduce a class of one dimensional interacting particle systems with *asymmetric rate functions*, i.e., the right hop rate $u_R(n)$ is an independent function, not just a constant multiple of the left hop rate $u_L(n)$. It is *a priori* not clear, whether a factorized steady state is at all possible for this asymmetric zero range process (AZRP). We derive a sufficient condition for AZRP to have a factorized steady state. Generalization of these asymmetric models to asymmetric misanthrope process (AMAP) and asymmetric finite range process (AFRP) are also investigated to find sufficient conditions on the rate functions that lead to factorized steady state in AMAP and cluster factorized form for AFRP. Interestingly, even though the steady state of both AZRP and AMAP are similar to that of ZRP, particle currents here show current-reversal as the density of the system is changed - a feature which can not be observed in ZRP with rates $u_R(n) = pu(n)$, $u_L(n) = qu(n)$. We also address the possibility of condensation transition in these systems and find that the onset of condensation can be tuned by the a factor that merely controls how often the particle chooses to move right, compared to its left hops.

The asymmetric hopping models which we discuss in this article are interesting in their own right. In addition, there are physical situations which may correspond to the

asymmetric diffusion proposed here. It is well known that geometry [29] or potential of mean forces [30] induce asymmetry across membrane channels and influence the particle fluxes across artificial or natural-biological pores. Such asymmetry is important for analyzing the dynamics of particle translocation [31] in biological channels. Also, this asymmetric diffusion effect may be utilized [32] to regulate transport and distribution of motile microorganisms in irregular confined environments, such as wet soil or biological tissues.

The article is organized as follows. In section 2, we introduce AZRP and derive the sufficient condition on the rate functions for obtaining a FSS. We then calculate the generic form of these rate functions $u_R(n), u_L(n)$ that gives rise to FSS, we devote the rest of the section for elaborate discussions on phenomena of condensation and current reversal. In section 3 we introduce asymmetric misanthrope process and show that the system can lead to FSS under certain conditions; current reversal and condensation phenomena in AMAP are discussed with specific examples. The most generic case, asymmetric finite range process (AFRP) is discussed in section 4, which leads to a cluster factorized steady states as in [10]. Finally, we summarize the results in section 5 with some discussions.

2. Asymmetric zero range process (AZRP)

2.1. The Model

Let us consider a system of N particles on a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \dots, L$. Each site i can accommodate $n_i \geq 0$ number of particles. The dynamics of the system is as follows. From a randomly chosen site i , having $n_i > 0$ particles, one particle is transferred either to the right neighbor $(i+1)$ with a rate $u_R(n_i)$ or to its left neighbor $(i-1)$ with a different rate function $u_L(n_i)$. Thus, the total number of particles $\sum_{i=1}^L n_i = N$ or the density $\rho = N/L$ is conserved. This stochastic process is a zero range process with asymmetric rate functions and hereafter we refer to it in short as asymmetric zero range process (AZRP). Clearly, in AZRP, particles at any given lattice site interacts with other particles at the same site through the hop rates which explicitly depend on the occupation number; interaction between particles at different sites is invoked only via the global conservation of N . In the following we show that this interacting particle system can have a factorized steady state if the rate functions satisfy certain constraints.

A special case of the model with $u_R(n_i) = pu(n)$, $u_L(n_i) = qu(n)$ is the well known zero range process [7] which describe symmetric (when $p = q$) or asymmetric (when $p \neq q$) transfer of particles. In this case, the steady state has a factorized form for any choice of rate function $u(n)$, and for arbitrary values of p, q

$$P_N(\{n_i\}) \sim \prod_{i=1}^L f(n_i) \delta\left(\sum_{i=1}^L n_i - N\right), \quad (1)$$

where $f(n) = \prod_{m=1}^n u(m)^{-1}$. We now ask, if such a factorized form is possible when rate functions for right and left hops are different, i.e., $u_R(n)$ and $u_L(n)$ have distinct functional forms. The master equation for AZRP is

$$\begin{aligned} \frac{d}{dt}P(\{n_i\}) &= \sum_{i=1}^L [u_R(n_i) + u_L(n_i)] P(n_1, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_L) \\ &- \sum_{i=1}^L [u_R(n_{i-1} + 1)P(\dots n_{i-1} + 1, n_i - 1, n_{i+1} \dots) \\ &+ u_L(n_{i+1} + 1)P(\dots n_{i-1}, n_i - 1, n_{i+1} + 1 \dots)] \end{aligned} \quad (2)$$

which governs how the probability $P(\{n_i\})$ of configuration $\{n_i\}$ evolves with time. Let us assume that the steady state of AZRP has a factorized form, as in Eq. (1)- then we use the FSS in Eq. (2) to check whether the steady state condition $\frac{d}{dt}P(\{n_i\}) = 0$ is satisfied automatically or does it put some constraint on $u_{R,L}(n)$ for which FSS is possible. With a FSS, the steady state master equation for any arbitrary configuration of AZRP reads as,

$$\begin{aligned} \sum_{i=1}^L [u_R(n_i) + u_L(n_i)] f(n_1) \dots f(n_{i-1}) f(n_i) f(n_{i+1}) \dots f(n_L) \\ - [\sum_{i=1}^L u_R(n_{i-1} + 1) \dots f(n_{i-1} + 1) f(n_i - 1) \dots \\ + \sum_{i=1}^L u_L(n_{i+1} + 1) \dots f(n_i - 1) f(n_{i+1} + 1) \dots] = 0. \end{aligned} \quad (3)$$

Now by shifting the index $i \rightarrow (i - 1)$ in the last sum we get an equation $\sum_{i=1}^L F(n_{i-1}, n_i) = 0$, where

$$\begin{aligned} F(m, n) &= u_R(n) + u_L(n) - u_R(m + 1) \frac{f(m + 1)f(n - 1)}{f(m)f(n)} \\ &- u_L(n + 1) \frac{f(m - 1)f(n + 1)}{f(m)f(n)}. \end{aligned} \quad (4)$$

Clearly we have a stationary measure if we can construct a single site function $h(n)$ that satisfy $F(m, n) = h(m) - h(n)$. Existence of such a function $h(n)$ ensures that $\sum_{i=1}^L F(n_{i-1}, n_i) = 0$ and thereby guarantees a factorized stationary measure. Since m, n are non-negative integers, let us first find what restrictions are imposed on $h(\cdot)$ from the boundary values. When $m = 0 = n$, from Eq. (4) we have $F(0, 0) = 0$, as $u_{R,L}(0) = 0$ (particle hopping is prohibited if the departure site is vacant) and $f(-1) = 0$ (a boundary condition that assigns zero weight for configuration having -ve occupation numbers); thus $F(m, n) = h(m) - h(n)$ is automatically satisfied. For other cases,

$$\begin{aligned} n = 0, m > 0 : & -u_L(1) \frac{f(m - 1)f(1)}{f(m)f(0)} = h(m) - h(0) \\ n > 0, m = 0 : & u_R(n) + u_L(n) - u_R(1) \frac{f(n - 1)f(1)}{f(n)f(0)} = h(0) - h(n). \end{aligned} \quad (5)$$

These equations are consistent if

$$f(n) = \frac{f(1)}{f(0)} \left[\frac{u_R(1) + u_L(1)}{u_R(n) + u_L(n)} \right] f(n - 1), \text{ and } h(n) = h(0) - u_L(1) \frac{f(n - 1)f(1)}{f(n)f(0)}. \quad (6)$$

Finally, a factorized steady state will be guaranteed if the above expressions of $h(n)$ and $f(n)$ consistently satisfy $F(m, n) = h(m) - h(n)$ for all $m > 0, n > 0$. This requirement actually constraints the right and left hop rates $u_{R,L}(n)$ to satisfy the following condition (from Eqs. (4) and (6)) ,

$$\frac{u_L(n+1)u_R(1) - u_R(n+1)u_L(1)}{[u_R(n) + u_L(n)][u_R(n+1) + u_L(n+1)]} = C, \quad (7)$$

where C is a constant independent of n . This completes the proof: *AZRP has a factorized steady state if the hop rates $u_{R,L}(n)$ satisfy Eq. (7)*. The weight factors $f(n)$ can be calculated from the recursion relation Eq. (6)

$$f(n) = [f(1)v(1)]^n \prod_{m=1}^n \frac{1}{v(m)} ; \text{ where } v(m) = u_R(m) + u_L(m), \quad (8)$$

where we set $f(0) = 1$, without loss of generality. Note a striking similarity of the weight factor $f(n)$ in AZRP with that of the ZRP. In Eq. (8) if one sets $f(1) = \frac{1}{v(1)}$, then the steady state of AZRP with specified hop rates $u_{R,L}(n)$ which satisfy Eq. (7) is exactly the same as that of the ordinary ZRP with hop rate $u_R(n) + u_L(n)$.

Note that, although validity of Eq. (7) is sufficient for AZRP to have a FSS, it is not *a priori* clear if there exists any such rate functions which satisfy this condition. To obtain a desired FSS as in Eq. (1) where

$$f(n) = \prod_{m=1}^n \frac{1}{v(m)} \text{ alongwith } f(0) = 1 \quad (9)$$

one can show, following Eqs. (8) and (7), that the asymmetric rate functions have the following generic functional form for $n \geq 1$,

$$u_R(n) = v(n) [\delta - \gamma v(n-1)] ; u_L(n) = v(n) [1 - \delta + \gamma v(n-1)]. \quad (10)$$

Clearly for $n = 0$, $u_R(0) = 0 = u_L(0)$ meaning $v(0) = 0$. Also we have set $\frac{C}{v(1)} = \gamma$. Now we have a family of asymmetric hop rates, characterized by two independent parameters $0 \leq \delta \leq 1$ and $0 \leq \gamma \leq \delta/v(n)|_{max} \ddagger$, which gives rise to a unique invariant measure described by Eqs. (1) and (9).

Some specific examples of AZRP will be discussed in the following sections. A simple situation is when $\gamma = 0$, where $u_R(n) = \delta v(n)$ and $u_L(n) = (1 - \delta)v(n)$. Since $\delta < 1$, the model is identical to an ordinary ZRP where particle chooses the right (or the left) neighbor as a target site with probability δ (or $1 - \delta$) and then hops to that site with rate $v(n)$. Obviously, $\delta = 0, 1$ corresponds to the usual ZRP where particles hop along a unique direction.

For any conserved system (N particles in L sites) with a factorized steady state

$$P_N(\{n_i\}) = \frac{1}{Q_N^L} \prod_{i=1}^L f(n_i) \delta(\sum_{i=1}^L n_i - N), \text{ with } f(n) = \prod_{m=1}^n \frac{1}{v(m)}, \quad (11)$$

$$\text{where } Q_N^L = \sum_{\{n_i\}} \prod_{i=1}^L f(n_i) \delta(\sum_{i=1}^L n_i - N) \quad (12)$$

\ddagger The range of δ and γ are fixed by the condition that the rates $u_{R,L}(n)$ must be positive.

is the canonical partition function, one can calculate the steady state average of any local observable straightforwardly. For completeness let us describe the procedure briefly. The grand partition function of the system is

$$Z_L(z) = \sum_{N=0}^{\infty} Q_N^L z^N = F(z)^L; \quad F(z) = \sum_{n=0}^{\infty} f(n) z^n, \quad (13)$$

where the fugacity z controls the average density of the system $\rho(z) = zF'(z)/F(z)$. The steady state average value of any local observable $O(n_i)$ is then

$$\langle O \rangle = \frac{1}{F(z)} \sum_{n=0}^{\infty} O(n) f(n) z^n, \quad (14)$$

which is a function of z . One can get the corresponding value for the conserved system with a given density $\rho = \rho^*$ by setting z to a specific value z^* which satisfy $\rho(z^*) = \rho^*$.

2.2. Condensation

The most interesting thing that happens in ZRP with a hop rate $v(n)$, or for any other model which has a factorized steady state given by Eq. (11), is the condensation transition. If the asymptotic form of $v(n)$ is

$$v(n) = v(\infty) \left(1 + \frac{b}{n^\sigma} + \dots \right), \quad (15)$$

condensation occurs for large densities either when $\sigma < 1$, or when $\sigma = 1$ and $b > 2$ [7]. It turns out that higher order terms in the series expansion are irrelevant in deciding the possibility of a condensation transition; they only play a role in determining the exact critical density above which the system forms a condensate. Since there are many exclusion models that have exact or approximate ZRP correspondence, the above criteria is extensively used for determining the possibility of phase separation transition [8]. A particularly simple case of (15), which is exactly solvable [7], is

$$v(n) = 1 + \frac{b}{n} \quad (16)$$

that results in a condensation transition for $b > 2$, when density ρ of the system crosses a critical value $\rho_c = \frac{1}{b-2}$.

In AZRP, to have a FSS given by (11) with $v(n) = 1 + \frac{b}{n}$ for $n \geq 1$ ($v(0) = 0$ by definition as already mentioned) the rate functions must follow Eq. (10). For this choice of $v(n)$, the model has three parameters $b > 0$, $0 < \delta \leq 1$ and γ ; here γ must be in the range $0 \leq \gamma \leq \frac{\delta}{v(n)|_{max}} = \frac{\delta}{1+b}$, so that the rates in Eq. (10) remain positive for all $n > 0$. Let us parametrize (b, δ, γ) in terms of three other parameters (b_R, b_L, α) as follows,

$$b = \alpha b_R + \bar{\alpha} b_L; \quad \delta = \alpha \left(2 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L} \right); \quad \gamma = \alpha \left(1 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L} \right), \quad (17)$$

where we use $\bar{\alpha} \equiv 1 - \alpha$ for notational convenience. The purpose of such parametrization will become clear in a moment. With these new parameters the hop rates of the model for the choice $v(n) = 1 + \frac{b}{n}$ can be written (using Eq. (10)) as

$$u_R(n) = \alpha \tilde{u}_R(n), u_L(n) = \bar{\alpha} \tilde{u}_L(n) \quad (18)$$

where for $n = 1$,

$$\begin{aligned}\tilde{u}_R(1) &= \left(2 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L}\right) [1 + \alpha b_R + \bar{\alpha} b_L] \\ \tilde{u}_L(1) &= \left(1 - \frac{b_R}{\alpha b_R + \bar{\alpha} b_L}\right) [1 + \alpha b_R + \bar{\alpha} b_L]\end{aligned}\quad (19)$$

and for $n > 1$

$$\begin{aligned}\tilde{u}_R(n) &= \left(1 + \frac{\alpha b_R + \bar{\alpha} b_L}{n}\right) \left[1 - \bar{\alpha} \frac{b_L - b_R}{n-1}\right] \\ \tilde{u}_L(n) &= \left(1 + \frac{\alpha b_R + \bar{\alpha} b_L}{n}\right) \left[1 + \alpha \frac{b_L - b_R}{n-1}\right].\end{aligned}\quad (20)$$

It is easy to see that the asymptotic forms of $\tilde{u}_{R,L}(n)$ are

$$\tilde{u}_R(n) = 1 + \frac{b_R}{n} + \dots; \quad \tilde{u}_L(n) = 1 + \frac{b_L}{n} + \dots \quad (21)$$

The new parameters α, b_R, b_L are all familiar to us: $b_{R,L}$, are coefficients of $\frac{1}{n}$ in the asymptotic expansion of the rates $\tilde{u}_{R,L}(\cdot)$ which normally take part in determining possibility of a condensation transition, and α may be considered as the probability that a particle chooses the right neighbor as the target site (note that $\alpha = \gamma - \delta$ varies in the range $(0, 1)$ for any $b > 0$). Thus, for the model in hand, particles choose to move right (or left) with probability α (or $1 - \alpha$) and hop there with rate $\tilde{u}_{R,L}(\cdot)$ respectively.

For $\alpha = 0$, particles in this model move only to left with rate $\tilde{u}_L(n) = 1 + \frac{b_L}{n}$ leading to a factorized steady state and a condensation for large densities when $b_L > 2$. Similarly for $\alpha = 1$, condensation occurs for $b_R > 2$. It is interesting to ask, ‘for a given fixed $b_{R,L}$, is it possible to observe a condensation transition by changing α ?’ Note that α determines how often the system chooses to hop right and a condensation transition, if appears by tuning only α , is exciting as it has not been observed earlier in ZRP or related models.

The difficulty, however, lies with the fact that for any given $b_{R,L}$ we do not have exact steady state measure (within this formalism) for all $\alpha \in (0, 1)$. The constraint comes from the requirement that the rate functions obtained in Eq. (18)-(20) must be positive valued for $n > 0$, which in turn restricts the value of α for which one can obtain the steady state weights exactly. In other words, for some $b_{R,L}$, it may not be possible to find $u_{R,L}(n)$ for which the steady state is factorized for any arbitrary $0 \leq \alpha \leq 1$. When both b_R and b_L are larger than 2, we have $b = \alpha b_R + (1 - \alpha) b_L > 2$; this case is not interesting because, even if we find suitable hop rates that describe this situation, and result in a FSS as in Eq. (11) with $v(n) = 1 + \frac{b}{n}$, the system will remain in the condensate phase for all α . Similarly, for $b_R < 2, b_L < 2$, condensation transition is not possible as b is smaller than 2 for any $0 < \alpha < 1$. Thus, we focus on the case where $b_R < 2$ and $b_L > 2$ (the other alternative $b_R > 2$ and $b_L < 2$ can be described in the same manner). For any fixed value of b_R the minimum and the maximum accessible values of α , for which one can have exact FSS with rate functions $u_{R,L}(n)$ given by Eq. (18)-(20) are respectively

$$\alpha_{min} = \max\left\{0, \frac{b_L - b_R - 1}{b_L - b_R}\right\}; \quad \alpha_{max} = \min\left\{1, \frac{1}{2} \frac{b_L}{b_L - b_R}\right\}. \quad (22)$$

These conditions on α are calculated simply by demanding positivity of the hop rates in (18)-(20).

To demonstrate the possibility of a condensation transition tuned by α , we consider AZRP with hop rates $u_{R,L}(n)$ given by (18)-(20), in two separate cases $b_R = \frac{3}{2}$ and $\frac{1}{2}$. The maximum and minimum values of α now depends on b_L ; in Fig. 1(a) and (b) we

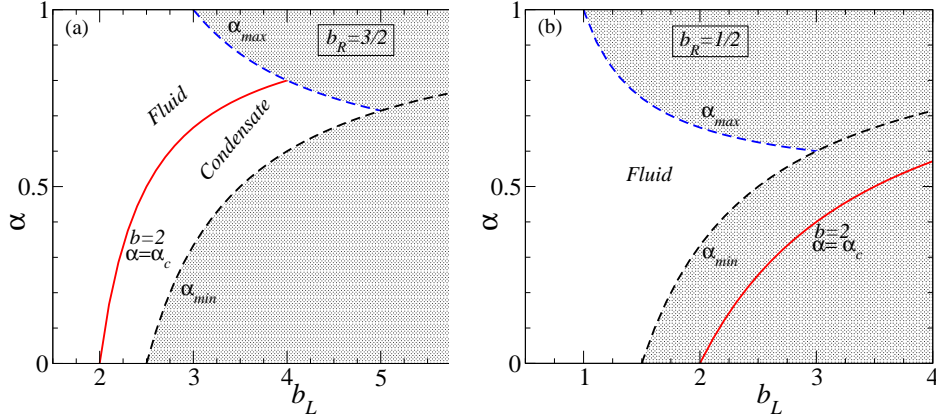


Figure 1. Condensation transition for AZRP dynamics given by Eqs. (18)-(20). For any given b_R, b_L the steady state has a factorized form when $\alpha \in (\alpha_{min}, \alpha_{max})$. Plots of α_{min} and α_{max} as function of b_L ($> b_R$) are shown here for (a) $b_R = \frac{3}{2}$ and (b) $b_R = 1/2$; we do not have exact steady state solution in the shaded regions where $\alpha > \alpha_{max}$ or $\alpha < \alpha_{min}$. Condensation transition occurs for large densities when $b = \alpha b_L + (1 - \alpha)b_R$ is larger than 2. In (a), this transition line $b = 2$, which separates the fluid phase from the condensate one, lies in the region where we have exact (factorized) steady state.

have plotted α_{min} and α_{max} in dashed lines for $b_R = \frac{3}{2}$ and $\frac{1}{2}$ respectively. The regions for $\alpha > \alpha_{max}$ and $\alpha < \alpha_{min}$ are shaded to indicate that within this formalism § the steady state does not have a factorized form in these regions. In the rest of regions, we have a factorized steady state given by Eqs. (11) and (16) and a condensation transition occurs here for large densities ($\rho > \frac{1}{b-2}$) when b is greater than 2, which corresponds to $\alpha > \alpha_c$ where

$$\alpha_c = \frac{b_L - 2}{b_L - b_R}. \quad (23)$$

In Fig. 1 we have also shown $\alpha = \alpha_c$ as a solid line, marked as $b = 2$ and correspondingly $\alpha = \alpha_c$. In the left panel ($b_R = \frac{3}{2}$) this line lies in the exactly solvable regime separating the fluid phase from the condensate one. For $b_R = 1/2$, we could not conclude if there is a condensation transition as the exact steady state measure in the neighborhood of $\alpha = \alpha_c$ line is not known. In fact, with some simple algebra one can show that for any $1 < b_R < 2$ the transition line lies in the exactly solvable regime, which is not the case when $0 < b_R \leq 1$.

As an explicit example, let us consider $b_R = \frac{3}{2}, b_L = \frac{9}{4}$; in this case clearly α can vary freely in the range $(0, 1)$, which can be seen from Fig. 1 (a). The rate functions,

§ Let us remind that the condition that we derive here is *not* a necessary but a sufficient condition.

from Eq. (18)-(20), are now $u_R(n) = \alpha \tilde{u}_R(n)$, $u_L(n) = (1 - \alpha) \tilde{u}_L(n)$ with

$$\tilde{u}_R(n) = \begin{cases} \frac{(13-3\alpha)(2-\alpha)}{2(3-\alpha)} & n = 1 \\ \frac{(4n-3\alpha+9)(4n+3\alpha-7)}{16n(n-1)} & n > 1 \end{cases} \quad \tilde{u}_L(n) = \begin{cases} \frac{(13-3\alpha)(3-2\alpha)}{4(3-\alpha)} & n = 1 \\ \frac{(4n-3\alpha+9)(4n+3\alpha-4)}{16n(n-1)} & n > 1 \end{cases}$$

It is easy to check that these functions result in the FSS given by Eq. (11) along with (16) where $b = \alpha b_R + (1 - \alpha) b_L$. For $\alpha = 1$, we have $b = b_R = \frac{3}{2}$ and the system remains in the fluid phase for all densities whereas for $\alpha = 0$, condensation occurs as $b = b_L = 9/4$. Interestingly for any arbitrary $0 < \alpha < 1$, $b = \frac{3}{4}(3 - \alpha)$ and a condensation transition takes place when α is decreased below $\alpha_c = \frac{1}{3}$ (from Eq. (23)). For any $\alpha > \alpha_c$, the system sets in the condensate phase only when the density of the system is increased above $\rho_c = \frac{4}{1-3\alpha}$.

2.3. Current reversal

Another interesting thing that happens in AZRP is the current reversal, where the direction of current depends on the particle density of the system. When AZRP with hop rates $u_{R,L}(n)$ has a factorized steady state given by Eq. (11) with $v(n) = u_R(n) + u_L(n)$, the steady state current in the system can be written as

$$J = \frac{1}{F(z)} \sum_{n=1}^{\infty} [u_R(n) - u_L(n)] f(n) z^n = \langle u_R(n) \rangle - \langle u_L(n) \rangle \quad (24)$$

where $F(z) = \sum_{n=0}^{\infty} z^n f(n)$. As we have discussed, a sufficient condition required for having a factorized steady state in AZRP is that $u_{R,L}(n)$ must have a form given by Eq. (10), with some $0 \leq \delta \leq 1$ and $0 \leq \gamma \leq \delta/v(n)|_{max}$. Then $u_R(n) - u_L(n) = v(n)[2\delta - 1 - 2\gamma v(n-1)]$ and thus

$$\begin{aligned} J &= (2\delta - 1) \langle v(n) \rangle - 2\gamma \langle v(n)v(n-1) \rangle \\ &= (2\delta - 1)z - 2\gamma z^2. \end{aligned} \quad (25)$$

In the last step we used $v(n) = \frac{f(n-1)}{f(n)}$ to calculate $\langle v(n) \rangle = \frac{1}{F(z)} \sum_{n=1}^{\infty} v(n) f(n) z^n = z$ and similarly, $\langle v(n)v(n-1) \rangle = z^2$.

In a simple ZRP with hop rates $u_R(n) = \alpha v(n)$ and $u_L(n) = (1 - \alpha)v(n)$, which corresponds to the choice $\delta = \alpha, \gamma = 0$, Eq. (25) leads to $J = (2\alpha - 1)z$. Thus, in ZRP, the direction of current J *can not* be changed by changing the density ρ (or equivalently the fugacity z); the direction is fixed only by α , i.e., J is positive (or negative) when $\alpha > \frac{1}{2}$ ($\alpha < \frac{1}{2}$). The change of density can only increase or decrease the magnitude of current, it can not change the direction of the flow. But surprisingly density dependent current reversal is possible in AZRP: for a fixed $u_{R,L}(n)$ the direction of the current may get reversed when the density of the system is changed. It is clear from Eq. (24) that such a reversal is not possible when $u_R(n) - u_L(n)$ has the *same sign* for all $n > 0$. In the following, we illustrate with a simple example that direction of current can be tuned by the density, when $u_R(n) > u_L(n)$ for all n except $n = 1$ where $u_R(n) < u_L(n)$. To this end, we consider AZRP with rate functions

$$u_R(n) = \begin{cases} \delta & n = 1 \\ \alpha & n > 1 \end{cases} ; \quad u_L(n) = \begin{cases} 1 - \delta & n = 1 \\ 1 - \alpha & n > 1 \end{cases} , \quad (26)$$

which follow Eq. (10) with $\alpha = \delta - \gamma$ varying in the range $(0, 1)$ and $v(n) = 1 \ \forall n > 0$ (and $v(0) = 0$). In this model isolated particles hop with a different rate than the rest. We also consider $\alpha > \frac{1}{2}$ and $\delta < \frac{1}{2}$ so that isolated particles hop preferentially in a different direction (here towards left) compared to particles from sites having two or more particles which preferentially move towards right. In this case, the flow direction of current can depend on the density of the system. For very large density there are only few sites which contain isolated particles and the current is expected to be positive (towards right) whereas for very low density most particles are isolated and one expects a negative current. Let us see if the direction of the current can be reversed when the density ρ of the system falls below a critical threshold ρ^* .

Since, $v(n) = 1 \ \forall n > 0$, this dynamics results in a FSS with $f(n) = 1 \ \forall n \geq 0$. Correspondingly $F(z) = \frac{1}{1-z}$ and $\rho = zF'(z)/F(z) = \frac{z}{1-z}$, which in turn implies $z = \frac{\rho}{1+\rho}$. Thus the current, from Eq. (25), is

$$J = \frac{\rho}{(1+\rho)^2} [2\delta - 1 + \rho(2\alpha - 1)]. \quad (27)$$

Since $\alpha > \frac{1}{2}$, and $\delta < \frac{1}{2}$, the current J flows in the negative direction if density ρ falls below $\rho^* = \frac{1-2\delta}{2\alpha-1}$.

In fact, it is clear from (25) that density dependent current reversal is a generic feature of AZRP. For generic AZRP with rate functions represented by (10), current reversal is expected at fugacity $z^* = \frac{2\delta-1}{2\gamma}$. But the crucial point, one must keep in mind, is z^* must lie in the range $0 < z^* < v(\infty)$ so that $z(\rho^*) = z^*$ would solve for a physically realizable density $\rho^* > 0$.

It is worth mentioning that, at the point of reversal (z^* or equivalently ρ^*), the average current J is zero but the steady state of the system is far different from the equilibrium one which also is characterized by zero current. For the model we discussed here, one obtains equilibrium *only* for $\delta = \alpha = \frac{1}{2}$ whereas the point of reversal $\rho^* = \frac{1-2\delta}{2\alpha-1}$ has a finite value for any $(\alpha > 1/2, \delta < 1/2)$ which correspond to a non-equilibrium scenario as the detailed balance condition is violated.

3. Asymmetric misanthrope process (AMAP)

Misanthrope process (MAP) is an interacting particle system, where hop rate of particles depends on both, the occupation of departure site and the arrival site. In contrast to ZRP, here particles at the departure site not only interact with other particles there, they also explicitly interact with particles at the arrival site. This model can have a factorized steady state in $1d$ if the hop-rate satisfies certain condition; for a periodic lattice with L sites $i = 1, 2, \dots, L$, each site i containing n_i particles, if particles move to their right neighbor with rate $u(n_i, n_{i+1})$, the condition for having a FSS reads as [11],

$$u(m, n) = u(m+1, n-1) \frac{u(1, m)u(n, 0)}{u(m+1, 0)u(1, n-1)} + u(m, 0) - u(n, 0). \quad (28)$$

In this section we generalize the misanthrope process to include asymmetric rate functions $u_{R,L}(\cdot, *)$, where the subscripts R, L stands for right, left and the arguments “.”

and “*” correspond to occupation number of departure and arrival sites respectively. We ask if the steady state of this asymmetric misanthrope process (AMAP) can be factorized, and if so, what would be the corresponding condition on the hop-rates ?

3.1. The model and the criterion for FSS

Like AZRP, the present section deals with a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \dots, L$. Each site i contains $n_i (\geq 0)$ number of particles as earlier but the hop rates in AMAP depend not only on the occupancy of the departure site but also on the arrival site. More precisely, a particle from a randomly chosen site i , provided $n_i > 0$, can either hop to its right neighbor $(i + 1)$ with a rate $u_R(n_i, n_{i+1})$ or it can move to its left neighbor $(i - 1)$ with a rate $u_L(n_i, n_{i-1})$.

To study whether AMAP can have a FSS, as before, we start with a conjecture that the steady state has a factorized form $P(\{n_i\}) \sim \prod_{i=1}^L f(n_i) \delta(\sum_{i=1}^L n_i - N)$ and look for conditions on the rate functions that satisfy $\frac{d}{dt}P(\{n_i\}) = 0$ in steady state where $P(\{n_i\})$, the probability of each configuration $\{n_i\}$, follows the master equation

$$\begin{aligned} \frac{d}{dt}P(\{n_i\}) = & \sum_{i=1}^L [u_R(n_i, n_{i+1}) + u_L(n_i, n_{i-1})] \dots f(n_{i-1})f(n_i)f(n_{i+1}) \dots \\ & - \sum_{i=1}^L u_R(n_{i-1} + 1, n_i - 1) \dots f(n_{i-1} + 1)f(n_i - 1)f(n_{i+1}) \dots \\ & - \sum_{i=1}^L u_L(n_{i+1} + 1, n_i - 1) \dots f(n_{i-1})f(n_i - 1)f(n_{i+1} + 1) \dots \end{aligned}$$

Let us collect all the terms from the right hand side of the above equation that contain both n_i and n_{i-1} as arguments of rate functions, and write them as $h(n_{i-1}) - h(n_i)$, where function $h(\cdot)$ is yet to be determined,

$$\begin{aligned} & u_R(n_{i-1}, n_i) + u_L(n_{i-1}, n_i) - u_R(n_{i-1} + 1, n_i - 1) \frac{f(n_{i-1} + 1)f(n_i - 1)}{f(n_{i-1})f(n_i)} \\ & - u_L(n_i + 1, n_{i-1} - 1) \frac{f(n_{i-1} - 1)f(n_i + 1)}{f(n_{i-1})f(n_i)} = h(n_{i-1}) - h(n_i). \end{aligned} \quad (29)$$

Clearly, existence of a function $h(\cdot)$ ensures that $\frac{d}{dt}P(\{n_i\}) = \sum_i h(n_{i-1}) - h(n_i) = 0$. Now let us check for the boundary conditions, i.e. when either of n_i, n_{i-1} or both are zero. Equation (29) is automatically satisfied when $n_i = n_{i-1} = 0$. When $n_i = 0, n_{i-1} = m > 0$, we have

$$h(m) = u_R(m, 0) + u_L(m, 0) - u_L(1, m - 1) \frac{f(m - 1)}{f(m)} \quad (30)$$

Here we have used the facts that $u_{R,L}(0, *) = 0$ (particles can not hop from vacant sites), $f(-1) = 0$ as $n_i > 0$, $f(1)/f(0) = 1$ (without loss of generality) and $h(0) = 0$ as the function $h(\cdot)$ in Eq. (29) is defined up to an arbitrary additive constant. Similarly, $n_{i-1} = 0, n_i = m > 0$ results in

$$h(m) = u_R(1, m - 1) \frac{f(m - 1)}{f(m)}. \quad (31)$$

Solving the above two equations for $f(m)$ and $h(m)$, we obtain

$$h(m) = u_R(1, m - 1)w(m) ; f(m) = \frac{f(m - 1)}{w(m)} = f(0) \prod_{k=1}^m \frac{1}{w(k)} \quad (32)$$

$$\text{where } w(m) = \frac{u_R(m, 0) + u_L(m, 0)}{u_R(1, m-1) + u_L(1, m-1)}.$$

Clearly, for any given $u_{R,L}(n, m)$, the steady state of AMAP is same as that of a simple ZRP with hop rate $w(m) = \frac{u_R(m,0)+u_L(m,0)}{u_R(1,m-1)+u_L(1,m-1)}$; the function $w(m)$, however satisfies $w(1) = 1$ (from above definition). The ZRP correspondence is not surprising, as we know that a factorized steady state (11) of any model can always be obtained from a simple ZRP with hop rate $\frac{f(m-1)}{f(m)}$. Finally using $f(m)$ and $h(m)$ in Eq. (29) we get the following condition on hop rates that ensures a FSS in AMAP,

$$\begin{aligned} u_R(m, n) + u_L(n, m) = & \left[\frac{u_R(m+1, n-1)}{w(m+1)} - u_R(1, n-1) \right] w(n) + u_R(m, 0) \\ & + \left[\frac{u_L(n+1, m-1)}{w(n+1)} - u_L(1, m-1) \right] w(m) + u_L(n, 0). \end{aligned} \quad (33)$$

When particles move only to right, i.e. $u_L(., *) = 0$ and $u_R(., *) = u(., *)$ this equation reduces to the condition Eq. (28) required for the usual totally asymmetric misanthrope process to have an FSS. In summary, a stochastic process on a $1d$ periodic lattice where particles (without obeying hardcore exclusion) hop to right or left with different rate functions $u_{R,L}(m, n)$ that depend on the occupation numbers m and n of departure and arrival site respectively, has a factorized steady state, as in Eq. (11) if the rate functions obey Eq. (33).

Equation (33) is more complicated than that the corresponding condition (7) for AZRP. For AMAP with any given rate function $u_{R,L}(m, n)$ one can easily check if they obey Eq. (33), but obtaining a generic form of hop rates that satisfy this condition is rather difficult. In the following we consider a few special cases. A very special class, is the equilibrium AMAP. If rate functions are related as follows

$$u_L(m, n) = u_R(n+1, m-1) \frac{w(m)}{w(n+1)}, \quad (34)$$

they surely satisfy (33) required for having a FSS, at the same time they also obey the condition of detailed balance. Equation (34) clearly describes a class of generic equilibrium AMAP models in the sense that $u_R(n+1, m-1)$ can still be chosen freely. Another class of AMAP models that has factorized steady state is

$$u_R(m, n) = \delta u(m, n) + \gamma u(m, 0) u(1, n); \quad u_L(m, n) = \gamma u(m, 0) u(1, n). \quad (35)$$

These rates, when used in Eq. (33) result in Eq. (28), which is the condition required for an ordinary misanthrope process with hop rate $u(m, n)$ to have a FSS. Thus, Eq. (35) describes a family of models, parametrized by two positive constants δ, γ and a positive-valued function $u(m, n)$ with $u(0, n) = 0$. In this case detailed balance is not satisfied and this class of models lead to a unique non equilibrium steady state having a factorized form as in Eq. (11) with weight function,

$$f(m) = \prod_{k=1}^m \frac{u(k, 0)}{u(1, k-1)}. \quad (36)$$

In section 3.3 we discuss a specific model of AMAP where hop rates follow Eq. (34). In the following section, we consider a model which neither satisfies Eq. (34) nor Eq.

(35) but still leads to a factorized steady state and exhibit density dependent current reversal.

3.2. Current reversal in AMAP

Like AZRP, it is possible to reverse the direction of the average current J in AMAP, only by tuning the number density ρ . Let us consider the following rate functions,

$$u_R(m, n) = \begin{cases} p & n = 0 \\ p_1 & n > 0, \quad m = 1 \\ p_2 & n > 0, \quad m > 1 \end{cases} ; u_L(m, n) = \begin{cases} q & n = 0 \\ q_1 & n > 0, \quad m = 1 \\ q_2 & n > 0, \quad m > 1 \end{cases} \quad (37)$$

It is easy to check that the rates (37) satisfy the constraint (33) only if

$$q_2 = p_2 - q + q_1 + \frac{((p+q)q_1)}{(p_1+q_1)} - \frac{(p(p_1+q_1))}{(p+q)} \quad (38)$$

With this choice of q_2 we have a factorized steady state given by Eq. (11) where

$$f(n) = \begin{cases} 1 & n = 0, 1 \\ \alpha^{n-1} & n \geq 2 \end{cases} ; \alpha = \frac{p_1 + q_1}{p + q}. \quad (39)$$

It is interesting to note that the steady state weight does not depend on p_2 ; any value of p_2 generates the same steady state as long as q_2 defined in Eq. (38) is positive. One must also note that though the rates in this model obey the generic constraint (33), they do not satisfy detailed balance condition and are not in the form of Eq. (34), also do not fall in the special class of rates given by (35).

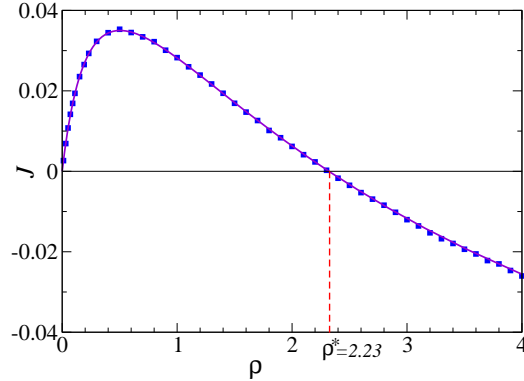


Figure 2. Current reversal in AMAP. Current J as a function of density ρ , measured from Monte Carlo simulation (symbols) of AMAP dynamics (37) with $(p = \frac{1}{2}, q = \frac{1}{4}, p_1 = \frac{1}{2}, q_1 = \frac{3}{4}, p_2 = 53/60, q_2 = 1)$ on a system of size L , is compared with exact results (lines) given by Eq. (42). As expected, current reversal occurs at density $\rho^* = 2.32$.

In the grand canonical ensemble, the partition function is $Z_L = F(z)^L$ with $F(z) = \sum_{n=0}^{\infty} f(n)z^n = \frac{1+(1-\alpha)z}{1-\alpha z}$, where the fugacity z lies in the range $(0, 1/\alpha)$, as

the radius of convergence of $F(z)$ is $z_c = 1/\alpha$. The density of the system is now

$$\rho(z) = z \frac{F'(z)}{F(z)} = \frac{z}{(1 - \alpha z)(1 + (1 - \alpha)z)} \quad (40)$$

$$\text{or } z = \frac{1 + \rho(2\alpha - 1) - \sqrt{(1 - \rho)^2 + 4\alpha\rho}}{2\rho\alpha(\alpha - 1)}. \quad (41)$$

The current in this system can be written as

$$\begin{aligned} J &= \frac{1}{F(z)^2} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [u_R(m, n) - u_L(m, n)] z^{m+n} f(m) f(n) \\ &= [(p - q) + (p_1 - q_1)z + (p_2 - q_2)(F(z) - z - 1)] \frac{F(z) - 1}{F(z)^2} \end{aligned} \quad (42)$$

If J needs to reverse its direction at some density ρ^* , the corresponding fugacity $z = z^*$ must be such that $J|_{z=z^*} = 0$; using Eq. (42) this leads to

$$z^* = \frac{1}{\alpha - 1} \left[1 - \sqrt{\frac{p(p_1 - p) + q(q_1 - q)}{p_1 q_1 - pq}} \right] \quad (43)$$

The above value of z^* will correspond to a feasible density only if $0 < z^* < 1/\alpha$; and then, one can obtain the corresponding density $\rho^* = \rho(z^*)$ using Eq. (40).

Now let us consider some specific cases, say $\alpha = \frac{5}{3}$. This may be obtained from, say, $(p = \frac{1}{2}, q = \frac{1}{4}, p_1 = \frac{1}{2}, q_1 = \frac{3}{4})$ with $q_2 = p_2 + \frac{7}{60}$ (from Eq. (38)). In this case $z_c = \frac{1}{\alpha} = \frac{3}{5}$ and the fugacity at the reversal point $z^* = \frac{3}{4}(2 - \sqrt{2}) < z_c$. So, for this choice of rates, current changes its direction when density of the system crosses a threshold value $\rho^* = \rho(z^*) = \frac{3}{7}(4 + \sqrt{2}) \approx 2.32$. In Fig. 2, we have shown a plot of the average current as a function of density; for very low density current flows towards right and increases as ρ is increased. Beyond a certain density where J reaches its maximum value, it decreases with ρ and finally starts flowing towards left as soon as the density becomes larger than $\rho^* \approx 2.32$.

Another interesting case is $\alpha = 1 = p + q$. In this case when $q_2 = p_2 + 1 - 2p_1$, we have a factorized steady state with a weight function $f(n) = 1 \forall n > 0$. Thus, $F(z) = \frac{1}{1-z}$, and $z = \frac{\rho}{1+\rho}$. Now current in the system, from Eq. (42),

$$J = \frac{\rho}{(1 + \rho)^2} [2p - 1 + (2p_1 - 1)\rho] \quad (44)$$

which changes its direction at $\rho^* = -\frac{2p-1}{2p_1-1}$. Thus reversal is possible at density $\rho = \rho^*$, when $p > \frac{1}{2}, p_1 < \frac{1}{2}$ or when $p < \frac{1}{2}, p_1 > \frac{1}{2}$. The noticeable point here is that the current in (44) is exactly similar to that of the AZRP current in (27) with $p \rightarrow \delta$ and $p_1 \rightarrow \alpha$, so is the point of reversal ρ^* ; but the the dynamics or AMAP is very different from that of AZRP. The similarity originates from the fact that the stationary state of both models are factorized with identical weight function $f(n) = 1 \forall n \geq 0$.

3.3. Condensation in AMAP

In this section, we turn our attention to AMAP models which give rise to condensation transition. A typical example of such asymmetric rate functions in AMAP that lead to

condensation is the following, where we consider rates $u_{R,L}(m, n)$ that fall in the special class of AMAP hop rates represented by Eq. (34) with $w(m) = \frac{1}{1+b}(1 + \frac{b}{m})$ (for $m \geq 1$),

$$u_L(m, n) = u_R(n+1, m-1) \frac{1 + \frac{b}{m}}{1 + \frac{b}{n+1}}. \quad (45)$$

This model would result in a FSS given by Eq. (11) along with the single site steady state weight

$$f(n) = \frac{n!(b+1)^n}{(b+1)_n}, \quad (46)$$

where $(c)_n = c(c+1) \dots (c+n-1)$ is the Pochhammer symbol. Now, we can calculate the grand canonical partition function $Z = F(z)^L$ where $F(z) = \sum_{n=0}^{\infty} \frac{n!(b+1)^n}{(b+1)_n} z^n$. Thus z varies in the range $(0, z_c)$ where $z_c = (1+b)^{-1}$ is the radius of convergence of $F(z)$. The density of the system is now $\rho(z) = z \frac{F'(z)}{F(z)}$; the critical density above which condensation takes place is

$$\rho_c = \rho(z_c) = \begin{cases} \infty & b \leq 2 \\ \frac{1}{b-2} & b > 2. \end{cases} \quad (47)$$

Thus, for AMAP with dynamics (45), the system under consideration can macroscopically distribute any number of particles if $b \leq 2$. However, for $b > 2$, the maximum allowed density is $\rho_c = \frac{1}{b-2}$ and if ρ is larger than ρ_c , a macroscopic number, $(\rho - \rho_c)L$, of particles gather on some particular site resulting in the formation of a single site condensate. So, like current reversal, condensation transition is also a common feature of both AZRP and AMAP.

4. Asymmetric finite range process process (AFRP)

Factorized steady state is a very special type of stationary measure but it is not a generic feature of the systems out of equilibrium. Stochastic processes like ZRP, AZRP, MAP, AMAP constitute a specific class of non-equilibrium processes that enjoy the simplicity of FSS. But one can also have pair factorized steady state (PFSS) [12] and cluster factorized steady state (CFSS) [10] for generic models where particle interaction extends beyond departure and arrival sites. Such finite range processes (FRP) introduce spatial correlations among occupation at different sites leading to *extended* condensates. Shape and size of the condensates spreading over a finite region in the space has been extensively studied in these systems [13]. In this section, we would like to focus on asymmetric FRP in $1d$ where the rate functions depend on occupation of K -nearest neighbors both to right and left of the departure site but the functional form of the hop rates now depend on the direction (left or right) of hopping. We would like to find out specific and sufficient conditions that must be obeyed by an asymmetric finite range process (AFRP) to achieve a cluster factorized steady state (CFSS).

4.1. The Model and criterion for CFSS

Consider a one dimensional periodic lattice with L sites labeled by $i = 1, 2, \dots, L$. Each site i contains an integer number of particles $n_i (\geq 0)$. A particle from a randomly chosen site i (with $n_i > 0$) can hop either to its nearest right neighbor ($i + 1$) with rate $u_R(n_{i-K}, n_{i-K+1}, \dots, n_i, n_{i+1}, \dots, n_{i+K-1}, n_{i+K})$ or it can hop to left nearest neighbor ($i - 1$) at a rate $u_L(n_{i-K}, \dots, n_{i-1}, n_i, \dots, n_{i+K})$. So both the right and left rate functions depend on $(2K + 1)$ terms, namely the departure site and its K nearest neighbors in both right and left directions. The $(2K + 1)$ arguments of $u_{R,L}(\dots)$ are spatially ordered, i.e. 1^{st} to $(2K + 1)^{th}$ arguments correspond to occupancy of site $i - K$ to $i + K$ respectively. Thus, $(K + 1)^{th}$ argument corresponds to the occupancy of the departure site i , and $(K + 2)^{th}$ and K^{th} arguments are the occupancy of the arrival site for right and left moves respectively. We assume that a cluster factorized steady state is possible for AFRP, as given below, and derive consistently the constraint required on the rate functions to obtain such a state. A cluster factorized steady state is represented by

$$P(\{n_i\}) \sim \prod_{i=1}^L g(n_i, n_{i+1}, \dots, n_{i+K}) \delta\left(\sum_{i=1}^L n_i - N\right), \quad (48)$$

where we call $g(\cdot)$ the cluster weight function that depends on $(K + 1)$ variables. In the steady state, with suitable rearrangement of terms, the master equation of AFRP can be written as a sum of L terms, each one being a unique function $F(\cdot)$ of $(2K + 3)$ arguments $(n_{i-K-1}, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_{i+K+1})$. So, in the steady state,

$$\frac{d}{dt} P(\{n_i\}) = \sum_{i=1}^L F(n_{i-K-1}, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_{i+K+1}) = 0. \quad (49)$$

A sufficient condition that satisfy the above equation (49) is when each of the L terms in the right hand side individually vanish, i.e. $F(n_{i-K-1}, \dots, n_{i-1}, n_i, n_{i+1}, \dots, n_{i+K+1}) = 0$ for every i ($i = 1, 2, \dots, L$). Clearly this condition is too restrictive and it is *not* a necessary condition for having CFSS. We restrict ourselves to this simple case which effectively leads to,

$$\begin{aligned} & u_R(n_{i-K}, \dots, n_i, n_{i+1}, \dots, n_{i+K}) + u_L(n_{i-K}, \dots, n_{i-1}, n_i, \dots, n_{i+K}) \\ &= u_R(n_{i-K-1}, \dots, n_{i-1} + 1, n_i - 1, \dots, n_{i+K-1}) \prod_{j=i-K-1}^i \frac{g(\tilde{n}_j, \tilde{n}_{j+1}, \dots, \tilde{n}_{j+K})}{g(n_j, n_{j+1}, \dots, n_{j+K})} \\ &+ u_L(n_{i-K+1}, \dots, n_i - 1, n_{i+1} + 1, \dots, n_{i+K+1}) \prod_{j=i-K}^{i+1} \frac{g(\hat{n}_j, \hat{n}_{j+1}, \dots, \hat{n}_{j+K})}{g(n_j, n_{j+1}, \dots, n_{j+K})} \end{aligned}$$

Here $\tilde{n}_j = n_j + \delta_{j,i-1} - \delta_{j,i}$ and $\hat{n}_j = n_j - \delta_{j,i} + \delta_{j,i+1}$. This constraint (50) on the rate functions can be satisfied by a family of hop rates, parametrized by $\delta > 0$ and $\gamma > 0$,

$$\begin{aligned} u_R(n_{i-K}, \dots, n_i, n_{i+1}, \dots, n_{i+K}) &= \delta \frac{g(n_{i-K}, n_{i-K+1}, \dots, n_i - 1)}{g(n_{i-K}, n_{i-K+1}, \dots, n_i)} \\ &\times \prod_{j=i-K+1}^i g(\hat{n}_j, \hat{n}_{j+1}, \dots, \hat{n}_{j+K}) + \gamma \prod_{j=i-K}^i \frac{g(\bar{n}_j, \bar{n}_{j+1}, \dots, \bar{n}_{j+K})}{g(n_j, n_{j+1}, \dots, n_{j+K})} \end{aligned}$$

$$u_L(n_{i-K}, \dots, n_{i-1}, n_i, \dots, n_{i+K}) = \delta \prod_{j=i-K}^{i-1} g(\tilde{n}_j, \tilde{n}_{j+1}, \dots, \tilde{n}_{j+K}) \frac{g(n_i - 1, n_{i+1} \dots n_{i+K})}{g(n_i, n_{i+1} \dots n_{i+K})}$$

where the newly introduced $\tilde{n}_j = n_j - \delta_{j,i}$ and δ, γ are constant parameters.

Let us consider the simplest case of AFRP, where particle interaction extends over a range $K = 1$. In this case, we expect a pair factorized steady state $P(\{n_i\}) \sim \prod_i g(n_i, n_{i+1}) \delta(\sum_{i=1}^L n_i - N)$ when hop rates are,

$$\begin{aligned} u_R(k, m, n) &= \frac{g(k, m-1)}{g(k, m)} \left[\delta g(m-1, n+1) + \gamma \frac{g(m-1, n)}{g(m, n)} \right] \\ u_L(k, m, n) &= \delta g(k+1, m-1) \frac{g(m-1, n)}{g(m, n)}. \end{aligned} \quad (50)$$

Note that for $\gamma = 0$, the hop rates satisfy detailed balance condition, and for $\gamma = 1, \delta = 0$, we recover the usual condition required for pair factorized state discussed in [12].

Also we observe that, current reversal is not possible for these particular set of rate functions in Eq. (50) which result in pair factorized steady states. This is because, the current in these models turns out to be $J = \gamma z$, which is just proportional to the fugacity z and since density $\rho(z)$ is a monotonic function of z , it is not possible to reverse the direction of the current by changing $z (\geq 0)$ or equivalently the density $\rho(z)$. In fact, for $K > 1$ also the rate functions in Eq. (50) give the same average current $J = \gamma z$, meaning that there is no current reversal by tuning of the fugacity or density for these class of models. However, the possibility of current reversal with a CFSS produced by asymmetric right-left rate functions in one dimension is still not ruled out, because, to satisfy the master equation in the steady state, one may find a balance condition different from the one used here; then J may not take such a simple form.

Another common feature of AZRP and AMAP is the formation of condensates which, unlike current reversal, can also be observed in case of AFRP within the framework of rate functions given by Eq. (50). We illustrate this briefly with a simple example. For $K = 1$, let us choose $g(m, n) = \frac{m+n+1}{(m+1)^b}$, where b is a tunable parameter indicating the onset of condensation. The corresponding right-left hop rates are

$$\begin{aligned} u_R(k, m, n) &= \frac{k+m}{k+m+1} \left[\delta \frac{m+n+1}{m^b} + \gamma \left(1 + \frac{1}{m}\right)^b \frac{m+n}{m+n+1} \right] \\ u_L(k, m, n) &= \delta \frac{k+m+1}{(k+2)^b} \left(1 + \frac{1}{m}\right)^b \frac{m+n}{m+n+1}. \end{aligned} \quad (51)$$

Using the transfer matrix formalism developed in [10], one can calculate the partition function $Q_L(z)$ in the grand canonical ensemble, where z is the fugacity associated with a particle in GCE and subsequently one can also obtain the density $\rho(z)$. Now if we proceed to calculate the critical density $\rho_c = \lim_{z \rightarrow 1} \rho(z)$, we find that for $b \leq 4$, ρ_c diverges indicating that the system remains in the fluid phase for $b \leq 4$ at any density. Whereas, when $b > 4$, we have a finite value of the critical density given by

$$\rho_c = \frac{\xi_1(b-1) - 2\xi_2(b) + \xi_3(b)}{2\xi_2(b) + 2\zeta(b-1)\sqrt{\xi_2(b)}} + \frac{\zeta(b-2) - \zeta(b-1)}{\sqrt{\xi_2(b)} + \zeta(b-1)}. \quad (52)$$

where $\xi_k(b) = \zeta(b)\zeta(b-k)$ and $\zeta(b)$ are Riemann zeta functions. So, for $b > 4$, if the density of the system is greater than the critical density *i.e.* $\rho > \rho_c$, one can observe a macroscopic number of particles $(\rho - \rho_c)L$ gathering at a single but arbitrary lattice site forming a single site condensate.

One can also observe spatially extended condensates in AFRP like the one discussed in [12], only this time with asymmetric rate functions given by

$$u_R(k, m, n) = \begin{cases} e^{U\delta_{m,1}}[e^{-J(n-m+3)} + e^{-2J\theta(m-n)} + e^{2J(1-\theta(m-n))}] & m \leq k, n+2 \\ e^{U\delta_{m,1}}[e^{-J(m-n-3)} + e^{2J}] & m > k, n+2 \\ e^{U\delta_{m,1}}[e^{-J(n-m+1)} + \theta(m-n) + e^{2J(1-\theta(m-n))}] & m > k, m \leq n+2 \\ e^{U\delta_{m,1}}[e^{-J(m-n-1)} + 1] & m \leq k, m > n+2 \end{cases}$$

and

$$u_L(k, m, n) = \begin{cases} e^{-J(k-m+3)+U\delta_{m,1}} & m \leq k+2, n \\ e^{-J(m-k-3)+U\delta_{m,1}} & m > k+2, n \\ e^{-J(k-m+1)+U\delta_{m,1}} & m \leq k+2, m > n \\ e^{-J(m-k-1)+U\delta_{m,1}} & m > k+2, m \leq n. \end{cases}$$

These rate functions lead to a PFSS with $g(m, n) = e^{-J|m-n|+\frac{U}{2}(\delta_{m,0}+\delta_{n,0})}$. Here J, U are the parameters that can be tuned to study the possibility of a condensation transition. As discussed in [12], when $J > J_c$, if the density ρ of the system is larger than the critical density $\rho_c = \frac{1}{e^{2(J-J_c)}-1}$ (where $J_c = U - \ln(e^U - 1)$), a macroscopic number of particles condensate over a spatial extent $O(L^{1/2})$ where L is the length of the lattice.

In brief, we have discussed the possibility of formation of both single site and extended condensates in case of AFRP with $K = 1$.

5. Summary

We have introduced a class of one dimensional stochastic models of interacting particles, without hardcore exclusion, where the particles are transferred asymmetrically to their neighbors: both right and left hop rates depend on the occupation of the departure site and their neighbors, but their functional forms are different. In usual driven diffusive systems the asymmetric rate appears from spatial inhomogeneity created by an external potential, which does not depend on the microscopic occupation. However it is not difficult to imagine, in fact actually has been shown recently, through simulations [29] and in biological systems [30, 32], that geometric irregularity can result in asymmetric diffusion of particles. It is interesting to ask what kind of rate functions are realistic for a particular geometry and the answer to this question is not understood well. In this article we focus on generic asymmetric rate functions and derive sufficient conditions on them for obtaining exact steady state measure for various asymmetric stochastic processes that include asymmetric zero-range process (AZRP), asymmetric misanthrope process (AMAP) and for the most generic case, asymmetric finite range process (AFRP).

Unlike ZRP, which has a factorized steady state (FSS) for any hop rate $u(n)$, AZRP with rate functions $u_{R,L}(n)$ lead to FSS when the rate-functions satisfy a specific

condition Eq. (7). On the other hand, a desired FSS as in Eq. (11) can *always* be obtained from a two parameter family of AZRP having left and right hop rates described by Eq. (10). It is well known [11, 10] that misanthrope process can not have a cluster-factorized steady state and its the steady state has a factorized form only for certain hop rates $u(m, n)$ which satisfy Eq. (28). AMAP shares the same feature but with a different constraint on the rate functions; it leads to a FSS only when the hop rates $u_{R,L}(m, n)$ follow Eq. (33). Both AZRP and AMAP show condensation transition, similar to other models having a FSS. Interestingly in case of AZRP, the condensation transition can be induced or broken by tuning the relative choice of $u_{R,L}(n)$ i.e. by changing the factor that decides how often a right move occurs with respect to a left move. The important role of asymmetric dynamics, both in AZRP and AMAP, appears in the particle current. Unlike ZRP or MAP where the direction of current is fixed by the external bias, here the direction can get reversed by changing particle density. We also extend this idea of asymmetry between right-left hop rates to obtain a cluster-factorized steady state in AFRP. In particular, we describe specific examples where the rate functions depend on the occupation of departure site and its two nearest neighbors (right and left), but the functional form for the right hop is different from that of the left; in this case we have obtained a sufficient condition required for a pair factorized state. Also, these examples include the formation of both localized and extended condensates. The general condition required for AFRP to have CFSS is much more complicated and we could not obtain the most generic class of rates which satisfy this constraint. However, in this article, we discuss a specific family of models parametrized by two constants although they do not show density dependent current reversal.

Some interesting open problems are AZRP, AMAP and AFRP with open boundaries or quenched disorder which may give rise to interesting boundary driven phase transitions. In this context, we should mention that site dependent current fluctuations above some critical current and that being indicator of condensation transition for open boundary ZRP with right-left rates related through a multiplicative constant has been studied in detail in [33]. One can also study the possibility of phase separation in exclusion models corresponding to the AZRP, AFRP dynamics studied here.

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